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Stochastic Modeling of a Billiard in a Gravitational Field: Power Law Behavior of Lyapunov Exponents

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We consider the motion of a point particle (billiard) in a uniform gravitational field constrained to move in a symmetric wedge-shaped region. The billiard is reflected at the wedge boundary. The phase space of the system naturally divides itself into two regions in which the tangent maps are respectively parabolic and hyperbolic. It is known that the system is integrable for two values of the wedge half-angle θ_1 and θ_2 and chaotic for $\theta_1 < \theta < \theta_2$. We study the system at three levels of approximation: first, where the deterministic dynamics is replaced by a random evolution; second, where, in addition, the tangent map in each region is replaced by its average; and third, where the tangent map is replaced by a single global average. We show that at all three levels the Lyapunov exponent exhibits power law behavior near θ_1 and θ_2 with exponents 1/2 and 1, respectively. We indicate the origin of the exponent 1, which has not been observed in unaccelerated billiards.

KEY WORDS: Lyapunov; scaling; chaos; nonlinear dynamics; billiard; gravity; random matrix.

1. INTRODUCTION

Recently, Benettin[1] has studied the behavior of ergodic billiards near parameter values for which the system is integrable. The boundary geometry depends continuously on the parameter. An example of such a system is the stadium, where the parameter ε is the length of the parallel edges. When $\varepsilon = 0$ the boundary is a circle and the billiard is integrable. For $\varepsilon > 0$, the system is known to be a K-system.[2,3] An interesting question concerns the asymptotic dependence of dynamical properties on

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the parameter as it vanishes. For a general class of two-dimensional billiards it is known from numerical studies that the maximal Lyapunov exponent scales as $\varepsilon^{1/2}$ as $\varepsilon \to 0$.[1] This result was recently proved by Wojtkowski.[4] In each case, for $\varepsilon > 0$, the action of the dynamics on the tangent space is either purely hyperbolic or hyperbolic and parabolic in disjoint regions. For $\varepsilon = 0$, the action is parabolic. Two remarkable properties of the scaling law are its apparent universality and robustness. For each billiard, both a Markovian model and a crude "mean field" approximation yield the same scaling relationship.

This paper concerns the dynamics of a two-dimensional billiard in a uniform gravitation field. Its principal properties were explored earlier by Lehtihet and Miller.[5] In common with the biliards mentioned above, the boundary depends on a parameter, but it is integrable for *two* of its values, θ_1 and θ_2. For $\theta_1 < \theta < \theta_2$ numerical experiments suggest K-system behavior with positive maximal Lyapunov exponents. Here, also, the phase space splits into two components, one where the tangent map is hyperbolic, and the other where it is parabolic. The hyperbolicity vanishes at θ_1 and θ_2. The maximal Lyapunov exponent scales differently at each integrable parameter value, as $|\theta - \theta_1|^{1/2}$ near θ_1, and as $|\theta - \theta_2|$ near θ_2.[5] In this study, the dynamical features which produce scaling are investigated. It is demonstrated that the scaling survives three levels of simplifying approximation which progressively ignore dynamial structure, thus illustrating the robustness of the scaling.

The construction of a rigorous proof of the existence of ergodic components in the phase plane and the scaling of Lyapunov exponents may be possible using the methods of Wojtkowski.[4] However, the inclusion of acceleration introduce the additional complicating feature of parabolic orbits. We hope to have more to say about this in the future. Here we try to understand the basic dynamical features which produce scaling of the Lyapunov exponent by studying a stochastic model at two levels.

We study the motion of a point particle in a uniform gravitational field constrained to move in a symmetric wedge-shaped region. The particle is reflected at the wedge boundary, that is, the component of the velocity tangential to the wedge boundary is preserved while the normal component is reflected. Between collisions with the boundary, the particle is uniformly accelerated parallel to the wedge bisector (see Fig. 1).

When the energy is fixed, the phase space for this system is three dimensional. If we consider the Poincaré section obtained by the return map to the wedge boundary, we obtain a two-dimensional system. The two dimensional phase space, which turns out to be a bounded region of the plane, can be divided into two disjoint regions (A and B). Region A consists of those phase points for which the next collision of the particle is with

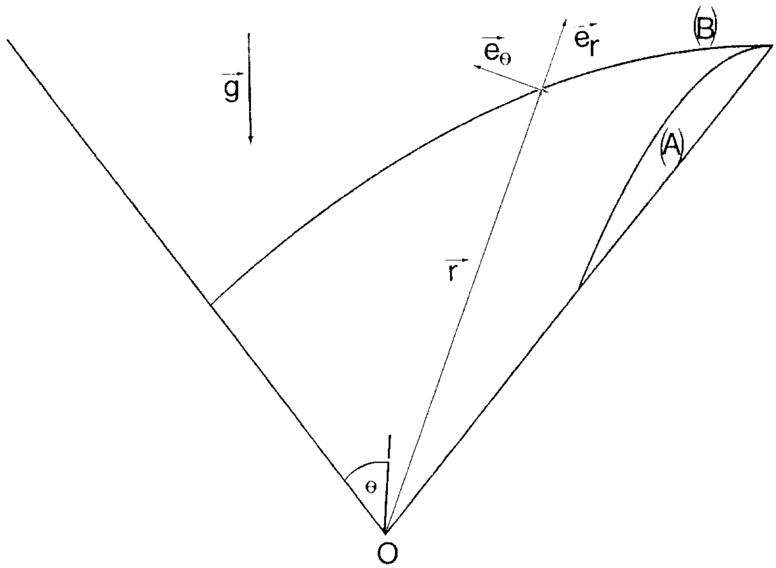

Fig. 1. Configuration space of the symmetric wedge. θ is the half-angle, $\mathbf{g}$ indicates the direction of acceleration, and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors for polar coordinates representing the billiard's position $\mathbf{r}$. The A (B) labels a trajectory which collides with the same (opposite) side.

the same side of the wedge (see Fig. 2). Region B is the complement of A and the vertex, i.e., it results in a collision with the opposite side. In a recollision, the normal component of the velocity is conserved, leading to a particularly simple form of the return map in region A. Similar simplifying features are absent in region B except at $\theta = \theta_1 = \pi/4$ and $\theta = \theta_2 = \pi/2$, where θ is the wedge half-angle. Consequently, the return map in the B region is highly nonlinear. It is possible to construct coordinate systems where both A and B return maps are area preserving.[5]

It can be shown that the system is integrable at θ_1 and θ_2. The earlier numerical study[5] demonstrated coexisting quasiperiodic and chaotic regions when $0 < \theta < \theta_1$ and total chaos when $\theta_1 < \theta < \theta_2$. In this paper we study the scaling properties of the maximal Lyapunov exponent[2] in the chaotic region using three levels of approximation. First, we replace the deterministic return map by a stochastic model where each successive point in the phase plane is selected at random. Second, in addition to selecting phase points at random, we replace the tangent map in each region by its respective regional average. Third, we average the tangent map over the whole phase space to obtain a single constant matrix.

Level one is investigated numerically. In the second level, in addition to the numerical study of scaling, bounds for the Lyapunov exponents are

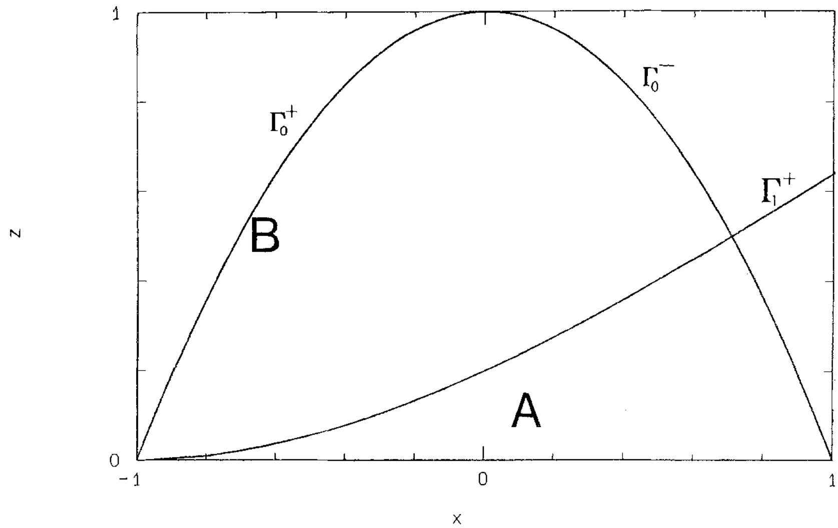

Fig. 2. The (x, z) phase space. A (B) labels the region from which trajectories collide with the same (opposite) side.

derived. In the third, the system is modeled by a single hyperbolic matrix. Both of the numerical studies, and the maximal eigenvalue of the hyperbolic matrix derived for level three, reproduce the same characteristic exponents as the original deterministic dynamical system. Upper and lower bounds at θ_2 and an upper bound at θ_1 also yield the same scaling, and the lower bound derived at θ_1 is consistent with the system behavior.

The surface of section and tangent map is defined in Section 2. Numerical experiments and results are described in Section 3. In Section 4 the average Jacobian matrices required for levels two and three are presented, and theorems required for establishing bounds on the Lyapunov exponent are proved.

2. DESCRIPTION OF THE MODEL

A convenient, area-preserving set of coordinates for the Poincaré surface of section is the component of the velocity tangent to the wedge boundary x and the square of the normal component z. The return maps for the A and B regions in this coordinate system, denoted hereafter by T_a and T_b, respectively, were derived by Lehtihet and Miller.[5]

The image of x_0, z_0 under T_a is

$$\begin{aligned} x_1 &= x_0 - 2 \cot \theta \sqrt{z_0} \\ z_1 &= z_0 \end{aligned} \tag{1}$$

whereas under T_b we find

$$\begin{aligned} x_1 &= (\sqrt{z_0} - \sqrt{z_1}) \operatorname{cor} \theta - x_0 \\ z_1 &= 2 \sin^2 \theta + 2\xi (\sqrt{z_0} - x_0 \tan \theta)^2 - z_0 \end{aligned} \tag{2}$$

where θ is the wedge half-angle, and $\xi = (1 - \tan^2 \theta)/(1 + \tan^2 \theta)^2$. The phase space is shown in Fig. 2 for a fixed value of the angle.

It is straightforward to compute the tangent map (Jacobian) for each region. In A

$$J_a = \begin{bmatrix} 1 & -\cot \theta/\sqrt{z} \\ 0 & 1 \end{bmatrix} \tag{3}$$

whereas in B,

$$J_b = \begin{bmatrix} -\dfrac{\cot \theta}{2\sqrt{z_1}} \dfrac{\partial z_1}{\partial x_0} - 1 & \dfrac{\cot \theta}{2\sqrt{z_0}} - \dfrac{\cot \theta}{2\sqrt{z_1}} \dfrac{\partial z_1}{\partial z_0} \\ -4\xi \tan \theta \, (\sqrt{z_0} - x_0 \tan \theta) & 2 \dfrac{\xi}{z_0} (\sqrt{z_0} - x_0 \tan \theta) - 1 \end{bmatrix} \tag{4}$$

Since we know[5] that $\sqrt{z_0} - x_0 \tan \theta > 0$ in the B region, and $\xi < 0$ for $\pi/4 < \theta < \pi/2$, it is easy to see that $\operatorname{Tr} J_b < -2$ and $\det J_b = 1$. This shows that J_b is reflection hyperbolic. It is clear that J_a is parabolic. This means that the action of J_a is to shear the tangent vector, while T_b acts hyperbolically.

We are interested in characterizing the chaotic behavior of the system. The fact that the action of J_b on the tangent space is hyperbolic when $\pi/4 < \theta < \pi/2$ gives us a clue as to why chaotic behavior is observed numerically, but it is by no means sufficient. The presence of asymptotic instability depends on how the tangent vector positions itself with respect to the local expanding direction along the orbit.

3. NUMERICAL RESULTS

Three numerical experiments were performed. First, the Lyapunov exponent was computed for the original dynamical system, then for a Bernoulli process defined on the same manifold with the same locally defined Jacobian matrix (described above as level one), and finally, for the two-state Bernoulli process (A or B) using the average Jacobian matrix for each region (level two), respectively. It is important to understand that, in common with billiards, the exponent is defined for the discrete time map [see (1) and (2)], and not the continuous time flow. Thus, λ is given by

$$\lambda_n = (1/n) \sum_1^n \ln |J_n t_n|, \qquad \lambda = \lim_{n \to \infty} \lambda_n$$

where J_n is the Jacobian matrix and t_n is a unit vector in the tangent space after n iterations.

In all cases the numerical calculations were carried out in double precision on an AT&T 3B2/400 computer running CFP (AT&T's floating point C compiler). The pseudo-random-number generator drand48( ), which is suplied with the compiler, was used to generate random numbers on the unit interval in levels one and two. The behavior of λ was examined at the following angles near 45°: 46°, 45.1°, 45.01°, 45.001°, 45,0001° and at 89°, 89.9°, 89.99°, and 89.999° near the integrable point at 90°. Successive values of λ_n were computed via a running average, i.e.,

$$\lambda_{n+1} = (n\lambda_n + \ln |J_{n+1} t_{n+1}|)/(n+1)$$

The values of λ_n were sampled every 10^5 iterations. With one exception, the criteria for termination was agreement between two successively sampled values (separated by 10^5 iterations) to within one part in 10^5. The length of the runs required to achieve this accuracy varied considerably. In the first case, where the true system dynamics was employed, correlations extended the duration considerably. However, in all cases, if θ was close to 90°, very long runs were required. This occurs because T_b, the hyperbolic portion of the transformation, is hardly ever invoked in the neighborhood of $\theta = 90°$, as the wedge is nearly flat and the billiard usually recollides on the same side. It is the essential reason that the Lyapunov number vanishes more rapidly near 90° than near 45°. The exception (see above) occurred at 89.999° for the true dynamics, where the program was stopped after 337,3000,000 iterations, as the value of λ_n was steady and the cpu time excessive.

It is necessary to keep in mind that the sequence λ_n does not converge uniformly. Because the process is probably a K system for the true dynamics, and definitely Bernoulli for the first two levels of approximation, the sequence $\{\lambda_n\}$ converges in the sense of the law of large numbers. Thus, agreement to five places between two successively sampled value of λ_n does not guarantee the validity of the first five figures of λ. Rather, they suggest the validity of approximately three figures or better. The choice of the sampling separation of 10^5 iterations, as well as the five-figure criteria for accuracy, were a compromise which was arrived at by experimentation. Shorter sampling times especially restulted in lack of satisfactory numerical reproducibility of the results due to the higher probability of a chance coincidence. Longer sampling separations and more restrictive criteria for termination simply required more cpu time than we could justify, and did not significantly alter the data.

Values of the computed Lyapunov number at each half-angle, $|\theta - \theta_i|$, $\log_{10}(\lambda)$, and $\Delta \log_{10}(\lambda)$, the difference between two successive values of

Table I. Lyapunov Exponents from Numerical Iteration for the Original Dynamical System[a]

θ	$\Delta\theta$	Lyapunov exponent	log (Lyapunov exponent)	Δ log (Lyapunov exponent)
46.0000	1.0	2.652504e-1	−0.576	0.480
45.1000	0.1	8.782750e-2	−1.056	0.505
45.0100	0.01	2.746759e-2	−1.561	0.515
45.0010	0.001	8.400783e-3	−2.076	0.500
45.0001	0.0001	2.656793e-3	−2.576	—
89.0000	1.0	2.684556e-2	−1.571	0.946
89.9000	0.1	3.038617e-3	−2.517	1.014
89.9900	0.01	2.945342e-4	−3.531	1.001
89.9990	0.001	2.9385e-5	−4.532	—

[a] $\Delta\theta$ is $|\theta-\theta_i|$ and $\Delta\log_{10}\theta$ is the decrement of $\log_{10}\theta$ for each successive pair of θ values.

$\log_{10}(\lambda)$, are listed in Table I–III for the true dynamical system, the level one stochastic approximation, and the level two approximation, respectively. Because each successive value of $|\theta-\theta_i|$ is reduced by a factor of 10, in the scaling region $\Delta\log_{10}(\lambda)$ should equal β, the characteristic exponent. The numerical values are consistent with $\beta=0.5$ at $\theta=45°$ and $\beta=1.0$ at $\theta=90°$.

Table II. Lyapunov Exponents from Numerical Iteration for the Case Where the Position in Phase Space is Chosen Completely at Random[a]

θ	$\Delta\theta$	Lyapunov exponent	log (Lyapunov exponent)	Δ log (Lyapunov exponent)
46.0000	1.0	2.83765e-1	−0.576	0.480
45.1000	0.1	9.542275e-2	−1.020	0.492
45.0100	0.01	3.078268e-2	−1.512	0.496
45.0010	0.001	9.808095e-3	−2.008	0.499
45.0001	0.0001	3.112281e-3	−2.507	—
89.0000	−1.0	3.005382e-2	−1.522	1.005
89.9000	−0.1	2.974898e-3	−2.527	1.001
89.9900	−0.01	2.965064e-4	−3.528	1.024
89.9990	−0.001	2.802369e-5	−4.552	—

[a] $\Delta\theta$ is $|\theta-\theta_i|$ and $\Delta\log_{10}\theta$ is the decrement of $\log_{10}\theta$ for each successive pair of θ values.

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Table III. Lyapunov Exponents from Numerical Iteration of the "Mean" Dynamical System[a]

θ	$\Delta\theta$	Lyapunov exponent	log (Lyapunov exponent)	Δ log (Lyapunov exponent)
46.0000	1.0000	2.423685e-1	−0.616	0.501
45.1000	0.1000	7.639950e-e	−1.117	0.500
45.0100	0.01	2.416116e-2	−1.617	0.500
45.0010	0.001	7.644548e-3	−2.117	0.499
45.0001	0.0001	2.419240e-3	−2.616	—
89.0000	−1.0	2.929141e-2	−1.533	1.001
89.9000	−0.1	2.922004e-3	−2.534	0.993
89.9900	−0.01	2.972168e-4	−3.527	1.026
89.9990	−0.001	2.801283e-5	−4.553	—

[a] The local Jacobian matrix is replaced by its average in region A or in region B. The $\Delta\theta$ is $|\theta - \theta_i|$ and $\Delta \log_{10} \theta$ is the decrement of $\log_{10} \theta$ for each successive pair of θ values.

4. THEOREMS AND COMPUTATIONS

Let μ_a and μ_b denote the measures of the A and B regions in the (x, z) coordinates. The average Jacobians in each region are denoted by X_a and X_b:

$$\begin{aligned} X_a &= (1/\mu_a) \int J(x, z)\, d\mu_a \\ X_b &= (1/\mu_b) \int J(x, z)\, d\mu_b \end{aligned} \tag{5}$$

where the integral of a matrix is taken to be the matrix obtained by integrating each element. Note that there is no guarantee that the average Jacobian X_b will be hyperbolic. With *some* effort, the integrals can be analytically evaluated and are given below:

$$X_a = \begin{bmatrix} 1 & -\dfrac{\pi/2 + \arcsin\{\alpha[(\alpha^2+1)(\alpha^2+4)]^{1/2}\}}{\alpha\mu_a} \\ 0 & 1 \end{bmatrix} \tag{6}$$

$$X_b = \begin{bmatrix} \dfrac{8\xi}{3\mu_b} - 1 & \dfrac{1}{\alpha}\left(\dfrac{\pi}{2} - \dfrac{\alpha^2-1}{\alpha^2+1} \arcsin \dfrac{\alpha^2-1}{\alpha^2+1}\right) \\ \dfrac{-\alpha\pi\xi}{\mu_b} & \dfrac{8\xi}{3\mu_b} - 1 \end{bmatrix} \tag{7}$$

where

$$\mu_b = (4/3)/(\alpha^2 + 1)^{1/2}, \qquad \mu_a = 4/3 - \mu_b \tag{8}$$

and $\alpha = \tan\theta$. We can easily check that X_b is hyperbolic and that $|\det X_b| > 1$. That is, X_b has two real eigenvalues λ_1 and λ_2, where $0 < |\lambda_1| < 1 < |\lambda_2|$.

In the level 2 mean field approximation the phase point moves independently between regions A and B; the probability of being in each region is proportional to its area. Thus, we have replaced the original dynamical system by (Ω, P, S), where $\Omega = \{a, b\}^Z$ is the set of infinite sequences in symbols a and b. We write $\omega = \{\omega_i: -\infty < i < \infty\} \in \Omega$ and we can interpret $\omega_0 = a$ or b to mean that the system starts in the A or B region, respectively. The evolution of the system is modeled by the shift map S. That is, $(S\omega)_i = \omega_{i+1}$. P is the Bernoulli product probability measure, which assigns $P(\omega_i = b) = \mu_b/(\mu_a + \mu_b) = p$.[6]

We are interested in obtaining information about the maximal Lyapunov exponent of the system as a measure of chaos. In level 2 the average action of the tangent map in the A and B regions is given by X_a and X_b, respectively. We derive bounds for the maximal Lyapunov exponent of independent products for a general class of matrices parametrized by ε, of which the X_a and X_b in our problem are particular examples. In our particular example, the parameter ε represents $\theta - \theta_1$ near θ_1, and $\theta_2 - \theta$ near θ_2.

The bounds are obtained by establishing the existence of an invariant cone. The existence of a family of invariant cones was used to obtain the rigorous results for planar billiards by Wojtkowski.[4]

Benettin[1] also obtained upper and lower bounds for the maximal Lyapunov exponent for products of 2×2 random matrices. He showed that the bounds are of the form $C\varepsilon^{1/2}$, where C is a constant, by using the existence of an invariant cone. His results were obtained without assuming independence. This was possible because, for the class of matrices he considered, the dilation of vectors in the invariant cone is bounded from below by $(1 + C\varepsilon^{1/2})$. In our model, the lower bound for the dilation in the cone is unity. Therefore, we need to known the asymptotic fraction of times the vectors in the cone spend away from the region where the dilation is too small to produce the expected scaling. This canot be assumed without some knowledge of the ergodic properties of the sequence of random matrices.

Let

$$X_b(\varepsilon) = \begin{bmatrix} a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & a(\varepsilon) \end{bmatrix}, \qquad \varepsilon > 0 \tag{9}$$

be a class of 2×2 real matrices with the following properties:

1. X_b has two distinct real eigenvalues, $0 < \lambda_1 \leqslant 1 \leqslant \lambda_2$.
2. $a(\varepsilon) = 1 + a_0 \varepsilon + O(\varepsilon)$, $a_0 > 0$.
3. $b, c \leqslant 0$.

It is easy to see that $\lambda_2 = a + (bc)^{1/2}$, $\lambda_1 = a - (bc)^{1/2}$, and the eigendirections for λ_2 and λ_1 have slopes $-(c/b)^{1/2}$ and $(c/b)^{1/2}$, respectively. Let

$$X_a(\varepsilon) = \begin{bmatrix} 1 & d(\varepsilon) \\ 0 & 1 \end{bmatrix}, \qquad \varepsilon \geqslant 0 \tag{10}$$

be a class of 2×2 real matrices, where $d(\varepsilon) \geqslant 0$.

Consider the product of identically distributed, matrix-valued, independent, Bernoulli random variables $\{X_i(\varepsilon)\colon 1 \leqslant i < \infty\}$. That is, consider products of the form $X_n(\varepsilon) \cdots X_2(\varepsilon) X_1(\varepsilon)$, where $X_i = X_a$ with probability $1 - p$, and X_b with probability p. Let

$$\lambda(\varepsilon) = \lim_{n \to \infty} (1/n) \ln \|X_n(\varepsilon) \cdots X_1(\varepsilon)\| \tag{11}$$

where $\|X\| = \sup\{|Xz|\colon z \in R^2, |z| = 1\}$, $|z|$ being the Euclidean norm. λ is called the maximal Lyapunov exponent and its existence in our case is guaranteed by Oseledec's multiplicative ergodic theorem.[7]

We first derive a formula for the lower bound of λ which will be used in the theorems that follow. Let $Z_0 = z$ be an arbitrary unit vector in R^2. Define $Z_n = [X_n \cdots X_1 z]/|X_n \cdots X_1 z|$, $n = 1, 2, \ldots$. Since $\{X_i\colon 1 \leqslant i < \infty\}$ is an independent sequence of random variables, $\{Z_n\}$ is a Markov chain, with the unit circle (S^1) as its state space. Note that the Markov chain $\{Z_n\}$ discribes the random evolution of the direction of a unit vector starting from z. The one-step transition probability $P(x, \cdot)$ for $\{Z_n\}$ is a discrete probability measure. For every $x \in S^1$, $P(x, \cdot)$ assigns a mass p to $X_b x/|X_b x|$ and a mass $1 - p$ to $X_a x/|X_a x|$. Let v be a stationary measure for the Markov chain, i.e., $\int v(dx)\, P(x, c) = v(c)$ for all measurable subsets c of S^1.[7] We observe that $\ln |X_n \cdots X_1 z| = \sum_{1 \leqslant i \leqslant n} \ln |X_i Z_{i-1}|$. It follows from Birkhoff's ergodic theorem that, for v a.e. $z \in S^1$,

$$\begin{aligned} &\lim_{n \to \infty} (1/n) \ln |X_n \cdots X_1 z| \\ &\quad = \lim_{n \to \infty} (1/n) \sum_{1 \leqslant i \leqslant n} \ln |X_i Z_{i-1}| \\ &\quad = p \int_{S^1} \ln |X_b y|\, dv(y) + (1 - p) \int_{S^1} \ln |X_a y|\, dv(y) \end{aligned} \tag{12}$$

Thus, we have

$$\lambda \geqslant p \int_{S^1} \ln |X_b y| \, d\nu(y) + (1-p) \int_{S^1} \ln |X_a y| \, d\nu(y) \tag{13}$$

where the inequality becomes an equality if there is a unique stationary measure for $\{Z_n\}$. We refer the reader to the paper by Furstenberg and Kifer[8] for the derivation of a general version of this result.

The product $X_n \cdots X_1$ leaves the cone σ_e formed by the expanding direction and the x axis invariant. Moreover, given a vector $v \in R^2$, $X_n \cdots X_1 v \in \sigma_e$ for some $n < \infty$ almost surely. From these two observations, we conclude that any stationary measure ν for $\{Z_n\}$ has to be supported on the arcs $\sigma = S^1 \cap \sigma_e$ (see Fig. 3). Therefore our formula for the lower bound becomes

$$\lambda \geqslant p \int_{\sigma} \ln |X_b y| \, d\nu(y) + (1-p) \int_{\sigma} \ln |X_a y| \, d\nu(y) \tag{14}$$

In Theorems 1 and 2 we choose matrix elements with similar asymptotic behavior to that encountered near θ_2 and θ_1, respectively. The only difference is that the matrices we consider for region B are simply

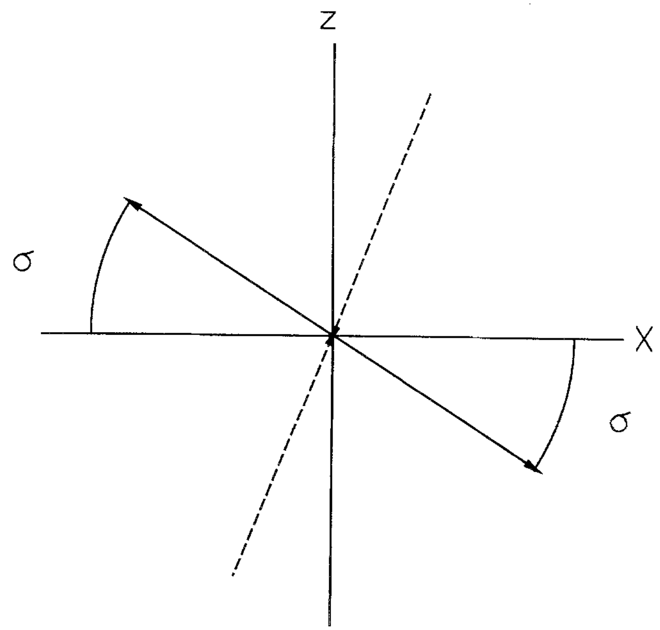

Fig. 3. Tangent space, showing the invariant cones.

hyperbolic, whereas the actual matrices are reflection hyperbolic. This is a minor difference, which does not affect the value of λ.

Theorem 1. Let $\lim_{\varepsilon\to 0} b(\varepsilon)=0$, $\lim_{\varepsilon\to 0} c(\varepsilon)=c_0<0$, $d(\varepsilon)=d_0\varepsilon+o(\varepsilon)$, $d_0<0$, $p(\varepsilon)=p_0\varepsilon+o(\varepsilon)$. Then $\lim_{\varepsilon\to 0}[\ln\lambda(\varepsilon)/\ln\varepsilon]=1$.

Proof. We obtain our result by establishing upper and lower bounds with appropriate asymptotic properties.

(i) *Upper bound*: Since $\|XY\|\leqslant\|X\|\,\|Y\|$, we have

$$\begin{aligned}(1/n)\ln\|X_n\cdots X_1\| &\leqslant (1/n)\sum_{1\leqslant i\leqslant n}\ln\|X_i\|\\ &=(n_a/n)\ln\|X_a\|+(n_b/n)\ln\|X_b\|\end{aligned}$$

where n_a and n_b are the numbers of X_a and X_b matrices in the string $X_n\cdots X_1$. From the law of large numbers it follows that as $n\to\infty$, $\lim(n_a/n)=1-p$ and $\lim(n_b/n)=p$. Therefore, from Eq. (11), we obtain

$$\lambda\leqslant p\ln\|X_b\|+(1-p)\ln\|X_a\| \tag{15}$$

With a little effort, it can be shown that as $\varepsilon\to 0$, $\infty>\lim\|X_b\|>1$ and

$$\|X_a(\varepsilon)\|\leqslant 1+|d(\varepsilon)|+d^2(\varepsilon)$$

From these observations it follows that there exists a $k_1>1$, $p_1>0$, $d_1>0$, and $\varepsilon_0>0$ such that if $\varepsilon<\varepsilon_0$, then $k_1>\|X_b\|>1$, $p(\varepsilon)\leqslant p_1\varepsilon$, $\|X_a\|\leqslant 1+d_1\varepsilon$. Thus,

$$\lambda\leqslant\varepsilon(p_1\ln k_1+d_1) \tag{16}$$

Lower Bound. Starting from Eq. (14), we construct a lower bound for λ by first showing that on a subset σ_1 of σ, as $\varepsilon\to 0$, $\lim[\ln X_b z|]>1$ and $v(\sigma_1)>0$, which imply $p\int\ln|X_b y|\,dv(y)>Cp$, where $C>0$. Thus, we will show that the lower bound scales like ε as $\varepsilon\to 0$.

As $\varepsilon\to 0$ the slope of the expanding direction approaches $-\infty$. Since X_b is limiting to a lower triangular matrix as $\varepsilon\to 0$, we can easily check that

$$\lim|X_b(\varepsilon)v|>[1+(c_0^2/2)]^{1/2}\qquad\text{for all}\quad v\in\sigma_1$$

where

$$\sigma_1=\{v\in\sigma\colon\ -2^{-1/2}\leqslant\text{slope of }v\leqslant 0\}$$

Therefore there exists an ε_1 such that if $\varepsilon < \varepsilon_1$, then

$$\ln|X_b z| > \ln[1 + (c_0^2/2)]^{1/2} > 0 \qquad \text{for all} \quad z \in \sigma_1 \tag{17}$$

We now estimate $\nu(\sigma_1)$. Since X_a is an upper triangular matrix and $d(\varepsilon) > 0$, X_a rotates vectors in the counterclockwise direction. We show that $(X_a)^n v \in \sigma_1$ for all $v \in \sigma$ if n is large enough. Let $m(v)$ be the slope of the vector v in R^2. Then

$$m[(X_a)^n v] = m(v)/[1 + ndm(v)] \tag{18}$$

From this it is clear that if $nd < -\sqrt{2}$, then $(X_a)^n v \in \sigma_1$ if $v \in \sigma$. Since $d(\varepsilon) = d_0 \varepsilon + o(\varepsilon)$, there exists an $\varepsilon_2 > 0$ such that if $\varepsilon < \varepsilon_2$, then $d(\varepsilon) < d_0 \varepsilon/2$. Let

$$n_0(\varepsilon) = \inf\{k \in N: k\,|d_0|\,\varepsilon > 2\sqrt{2}\}$$

Clearly, $(X_a)^{n_0} v \in \sigma_1$ for all $v \in \sigma$. Let $\gamma = 2\sqrt{2}/|d_0|$; then, as $\varepsilon \to 0$,

$$\lim[1 - p(\varepsilon)]^{\gamma/\varepsilon} = e^{-\gamma p_0} > 0$$

Therefore, there exists an $\varepsilon_3 > 0$ such that if $\varepsilon < \varepsilon_3$, then

$$[1 - p(\varepsilon)]^{n_0(\varepsilon)} > (e^{-\gamma p_0})/2 > 0$$

Since $(X_a)^{n_0} v \in \sigma_1$ for all $v \in \sigma$,

$$P^{n_0}(v, \sigma_1) > (e^{-\gamma p_0})/2 \qquad \text{for all} \quad v \in \sigma$$

where P^n is the n-step transition probability for the Markov chain $\{Z_n\}$.[6] Therefore,

$$\nu(\sigma_1) = \int P^{n_0}(x, \sigma_1)\, d\nu(x) > (e^{-\gamma p_0})/2 \tag{19}$$

Since $\ln|X_b z| > \ln[1 + c_0^2/2)]^{1/2}$, we have shown that

$$\int \ln|X_b y|\, d\nu(y) > \ln[1 + (c_0^2/2)]^{1/2}(e^{-\gamma p_0})/2 = L$$

for all $\varepsilon < \varepsilon'$, where $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Therefore, $\lambda \geqslant pL = Lp_0\varepsilon + o(\varepsilon)$. Q.E.D.

Theorem 2. As $\varepsilon \to 0$, let $\lim b(\varepsilon) = b_0 < 0$, $\lim d(\varepsilon) = d_0 < 0$, and $c(\varepsilon) = c_0\varepsilon + o(\varepsilon)$, $c_0 < 0$. Then there exist k_1 and $k_2 > 0$ such that

$$k_1\varepsilon + o(\varepsilon) \leqslant \lambda(\varepsilon) \leqslant k_2\varepsilon^{1/2} + o(\varepsilon^{1/2}) \tag{20}$$

Proof. We first observe that the slope of the expanding direction goes to zero as $\varepsilon \to 0$. Therefore the invariant cone σ collapses onto the x axis as $\varepsilon \to 0$. Let $\|X\|_\sigma = \sup\{|Xv|: v \in \sigma\}$. Next, we observe that, given $\varepsilon > 0$, for a.e. ω (sequence of X_i), there exists a $k = k(\varepsilon) \in N$ such that $X_k X_{k-1} \cdots X_1 v \in \sigma_\varepsilon$ for every $v \in R^2$. From this it follows that, if $n > k$, $\|X_n \cdots X_1\| \leqslant C \|X_n \cdots X_{k+1}\|_\sigma$, where $C > 0$ is a constant. Therefore

$$\begin{aligned}
\lim_{n\to\infty} (1/n) \ln \|X_n \cdots X_1\| &\leqslant \lim_{n\to\infty} (1/n) \ln \|X_n \cdots X_{k+1}\|\sigma + \lim_{n\to\infty} (1/n) \ln C \\
&= \lim_{n-k\to\infty} [1/(n-k)] \ln \|X_n \cdots X_{k+1}\|_\sigma \\
&= p \ln \|X_b\|_\sigma + (1-p) \ln \|X_a\|_\sigma
\end{aligned} \tag{21}$$

Since the slope of the expanding direction is

$$[c(\varepsilon)/b(\varepsilon)]^{1/2} = c_1 \varepsilon^{1/2} + o(\varepsilon^{1/2}), \qquad c_1 > 0 \tag{22}$$

it is easily verified that

$$\|X_a\|_\sigma \leqslant 1 + c_1 \varepsilon^{1/2} + o(\varepsilon^{1/2}) \tag{23}$$

and

$$\lambda_2(\varepsilon) = a(\varepsilon) + (bc)^{1/2} = 1 + c^2 \varepsilon^{1/2} + o(\varepsilon^{1/2}), \qquad c_2 > 0 \tag{24}$$

Therefore,

$$\|X_b\|_\sigma = 1 + c_2 \varepsilon^{1/2} + o(\varepsilon^{1/2}) \tag{25}$$

Since

$$\lambda \leqslant p \ln \|X_b\|_\sigma + (1-p) \ln \|X_a\|_\sigma \tag{26}$$

we have established the upper bound.

Lower Bound. It is easy to show that

$$\|X_b(1, 0)\|^2 = \lambda_1 \lambda_2 + (1/4) \sec^2 \varphi \, (\lambda_2 - \lambda_1)^2$$

where φ is the angle between the expanding direction and the x axis. From this we see that

$$\|X_b(1, 0)\| = 1 + k\varepsilon + o(\varepsilon), \qquad k > 0 \tag{27}$$

since $\|X_b v\| \geqslant \|X_b(1, 0)\|$ $\forall v \in \sigma$, we have

$$\int \ln \|X_b v\| \, d\nu(v) \geqslant \int \ln \|X_b(1, 0)\| \, d\nu(v) = \ln[1 + k\varepsilon + o(\varepsilon)] \qquad (28)$$

From this, the lower bound stated in the theorem folows. Q.E.D.

In section 2 it was pointed out that λ scales as $\varepsilon^{1/2}$ near θ_1. This is reflected in the behavior of the upper bund derived above. A more detailed investigation of ν is being developed which yields a more accurate lower bound which also scales as $\varepsilon^{1/2}$.

We conclude this section with a discussion of the level 3 approximation, which replaces the tangent map everywhere by a single matrix X, the average of the Jacobian matrix over the whole phase plane. The average matrix can be evaluated directly from Eqs. (6) and (7) using

$$X = (\mu_a X_a + \mu_b X_b)/(\mu_a + \mu_b) \qquad (29)$$

It is easy to show that $-X$ is hyperbolic with largest eigenvalue $\lambda_2 = 1 - 2\xi + (bc)^{1/2}$, where b and c are the off-diagonal elements of $-X$. Near $\theta = \theta_1$ and $\theta = \theta_2$ it follows that

$$\lambda_2 = 1 + k|\xi|^{1/2} + o(|\xi|^{1/2}), \qquad k > 0 \qquad (30)$$

Since

$$|\xi| = k_1(\theta - \theta_1) + o(\theta - \theta_1), \qquad k_1 > 0 \quad \text{as } \theta \to \theta_1$$

$$|\xi| = k_2(\theta - \theta_2)^2 + o[(\theta - \theta_2)^2], \qquad k_2 > 0 \quad \text{as } \theta \to \theta_2$$

we obtain the scaling predicted by the numerical experiments in all other levels.

5. CONCLUDING REMARKS

We have shown that the scaling property of the Lyapunov exponent near each parameter value at which the system is integrable is preserved by three levels of approximation. In levels 1 and 2, a stochastic process is substituted for the true system dynamics. Levels 2 and 3 may be thought of as man field theories, in which the tangent map is replaced by an appropriate average. Even though the crudest of the models, namely level 3, gives us the correct scaling behavior, we have quite a bit more to gain by considering level 2. Here we can clearly see that the power law scaling exponent of 1.0 is due to the fact that the region of the phase space with the hyperbolic tangent map (region B) vanishes as $\theta \to \theta_2$. This does not happen in any of the planar billiards considered by Benettin, where a power law exponent of 1/2 is observed. We have demonstrated that the mean field theory considered in this paper has enough structure, especially because of the fact that the tangent maps maintain their noncommutativity, to provide qualitative information concerning the behavior of Lyapunov exponents near values of a parameter where the system is integrable.

The robustness of the scaling laws demonstrated here suggests that the stable and unstable manifolds of the original system may not be tangent and thus may be worth investigating numerically.[9]

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